Categorified central extensions, étale Lie 2-groups and Lie's Third Theorem for locally exponential Lie algebras

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Abstract

Lie's Third Theorem, asserting that each finite-dimensional Lie algebra is the Lie algebra of a Lie group, fails in infinite dimensions. The modern account on this phenomenon is the integration problem for central extensions of infinite-dimensional Lie algebras.

This paper remedies the obstructions for integrating central extensions from Lie algebras to Lie groups by generalising central extensions. These generalised central extensions are categorified versions of ordinary central extensions of Lie groups, i.e., central extensions of Lie 2-groups. In this context, the integration problem can always be solved. The main application of this result is that a locally exponential Lie algebra (e.g., a Banach–Lie algebra) integrates to a Lie 2-group in the sense that there is a natural Lie functor from certain Lie 2-groups to Lie algebras, sending the integrating Lie 2-group to an isomorphic Lie algebra.

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Introduction

This paper sets out to resolve obstructions for integrating Lie algebras and central extensions of them in terms of Lie 2-groups. It is a celebrated theorem that each finite-dimensional Lie algebra is the Lie algebra of a Lie group, which is known as Lie's Third Theorem, which was proven by Lie in a local versions and in full strength by Élie Cartan (cf. [Car30] and references therein). It has also been Élie Cartan, who first remarked in [Car36] that one may also use the fact that $\pi_2(G)$ vanishes for any finite-dimensional Lie group[1] to prove Lie's Third Theorem. If G is infinite-dimensional, then $\pi_2(G)$ does not vanish any more, for instance for $C(S^1, \mathrm{SU}(2))$ or $PU(\mathcal{H})$, and van Est and Korthagen used this for $C(S^1, \mathrm{SU}(2))$ in [vEK64] to construct an example of a Banach–Lie algebra

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[1]originally, Cartan's condition was that the first two Betti numbers vanish

which cannot be the Lie algebra of a Lie group (cf. [DL66] for the corresponding construction for $PU(\mathcal{H})$).

However, there is a large class of infinite-dimensional Lie algebras, which integrate to a *local* Lie group, namely locally exponential Lie algebras. In particular, all Banach–Lie algebras belong to this class. The non-existence of a (global) Lie group, integrating a locally exponential Lie algebra may thus be regarded as the obstruction against the corresponding local Lie group to enlarge to a (global) Lie group. This is why a Lie algebra, which is the Lie algebra of a global Lie group is often called *enlargeable*, whilst a Lie algebra is called *integrable* if it is the Lie algebra of a local Lie group (cf. [Nee06]).

The most sophisticated tool for the analysis of enlargeability of locally exponential Lie algebras is Neeb's machinery for integrating central extensions of infinite-dimensional Lie groups, developed in [Nee02].If

$$\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$$

is a central extension of Lie algebras and G is a simply connected Lie group with Lie algebra $\mathfrak{g}$, then this central extension integrates to a central extension of Lie groups if and only if the period group $\mathrm{per}_{\widehat{\mathfrak{g}}}(\pi_2(G)) \subseteq \mathfrak{z}$ is discrete. This theory applies in particular to a exponential Lie algebra $\mathfrak{g}$, since then

$$\mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \to \mathfrak{g}_{\mathrm{ad}}$$

is a central extension and there always exist a simply connected Lie group G_{ad} with $L(G_{\mathrm{ad}}) = \mathfrak{g}_{\mathrm{ad}}$. Thus the obstruction for $\mathfrak{g}$ to be non-enlargeable is the non-discreetness of $\mathrm{per}_{\mathfrak{g}}(\pi_2(G_{\mathrm{ad}}))$. If $\mathfrak{g}$ is finite-dimensional, then $\pi_2(G_{\mathrm{ad}})$ vanishes and Lie's Third Theorem is immediate. From this point of view the theorem seems to be merely a homotopy-theoretic accident.

Enlarging local groups and integrating central extensions obey a common pattern. The obstruction for enlarging a local Lie group to a global one is an associativity constraint, which is coupled to topological properties of the local group (cf. [Smi50], [Smi51a], [Smi51b], [vE62a], [vE62b] and [BR08]). In general, global associativity can not be achieved, as the counterexamples, mentioned above, show. In the integration problem for cocycles, the obstruction for $\mathrm{per}_{\widehat{\mathfrak{g}}}(\pi_2(G)) \subseteq \mathfrak{z}$ to be discrete ensures that a cocycle condition holds for a certain universal integrating cocycle.

The upshot of this paper is that one may *relax* global associativity and cocycle conditions at the same time by introducing 2-groups. In these 2-groups, the corresponding equalities do not hold not exactly, but up to coherent correction terms. Motivated be this observation, our approach to Lie's Third Theorem is first to consider the integration of central extensions of Lie algebras to central extensions of Lie 2-groups. After having performed the integration theory for general central extensions we apply the results to the central extension $\mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \to \mathfrak{g}_{\mathrm{ad}}$ for $\mathfrak{g}$ locally exponential in order to obtain our version of Lie's Third Theorem.

The paper is organised as follows. In the first section we repeat the definitions of 2-groups and group cohomology with coefficients in abelian 2-groups. The second section concerns smooth structures on 2-groups and introduces the notions of Lie 2-groups. The third section presents the key ideas of the procedure for integrating central extensions of Lie algebras to central extensions of Lie 2-groups and shows that the this procedure is universal. The last section is devoted to set up the Lie theoretic basics for étale Lie 2-groups and to prove the following theorem.

Theorem. *Let $\mathfrak{g}$ be a locally convex Lie algebra and G be a locally convex Lie group with $L(G) = \mathfrak{g}$. Then each central extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ of Lie algebras integrates to a central extension of étale Lie 2-groups.* ∎

From this, Lie's Third Theorem is an immediate consequence.

Theorem. *Let $\mathfrak{g}$ be a locally exponential Lie algebra. Then there exists an étale Lie 2-group $\mathcal{G}$, such that $\mathcal{L}(\mathcal{G})$ is isomorphic to $\mathfrak{g}$.* ∎

In the end we indicate some directions for further research and give some elementary details on locally convex Lie groups in an appendix.

Acknowledgements

Conventions

For us, a manifold is a Hausdorff space, locally homeomorphic to open subsets of some locally convex space with diffeomorphic coordinate changes. A Lie group is a group, which is a manifold with smooth group operations (cf. Definition A.1 for details on this). For M, N pointed manifolds and $f : M \to N$, smooth and pointed, $df : T_*M \to T_*N$ denotes the differential in the base point and $T^2 f$ denotes the second order Taylor monomial

$$T_*M \to T_*N, \quad x \mapsto \frac{1}{2} d^2 f(x, x).$$

Unless stated otherwise, G shall always be a simply connected Lie group with Lie algebra $\mathfrak{g}$, which we usually identify with $T_e G$. Moreover, $\mathfrak{z}$ shall always denote a sequentially complete locally convex space.

If G is a group and A is a G-module (if the module structure is not mentioned explicitly it is either assumed to be natural or trivial), then

$$C^n(G, A) := \{f : G^n \to A | f(g_1, ..., g_n) = 0 \text{ if one } g_i = e\}$$

is the group of normalised A-valued n-cochains on G and we denote by

$$\mathtt{d}_{\mathrm{gp}} : C^*(G, A) \to C^{*+1}(G, A), \quad \mathtt{d}_{\mathrm{gp}} f(g_0, ...g_n) =$$

$$g_0.f(g_1, ..., g_n) - \sum_{i=0}^{n-1} (-1)^i f(g_0, ..., g_i g_{i+1}, ..., g_n) - (-1)^n f(g_0, ..., g_{n-1})$$

the ordinary group differential. As usual, $Z^n(G, A)$ and $B^n(G, A)$ denote the groups of n-cocycles and n-coboundaries and $H^n(G, Z)$ the corresponding cohomology group. If $f \in Z^2(G, Z)$, then

$$(a, g) \cdot (b, h) = (a + b + f(g, h), gh)$$

defines a group structure on $Z \times G$, which we denote by $Z \times_f G$.

We denote by $\Delta^{(n)} \subseteq \mathbb{R}^n$ the standard n-simplex. For an Hausdorff space X, $C_n(X) = \langle C(\Delta^{(n)}, X) \rangle$ denotes the group of singular n-chains in X and $\partial : C_*(X) \to C_{*-1}(X)$ the corresponding singular differential. As above, $Z_n(X)$ and $B_n(X)$ denote the corresponding cycles and boundaries and $H_n(X)$ the singular homology of X.

If $\mathcal{C}$ is a small category (denoted by a calligraphic letter), then $\mathcal{C}_0$ and $\mathcal{C}_1$ are the sets of objects and morphisms (denoted by roman letters). The structure maps of $\mathcal{C}$ are alwasy denoted by s, t, id and $\circ$. If $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a functor, then $\mathcal{F}_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $\mathcal{F}_1 : \mathcal{C}_1 \to \mathcal{D}_1$ are the corresponding maps on the set of objects and morphisms. Likewise, if $\alpha : \mathcal{F} \Rightarrow \mathcal{F}'$ is a natural transformation, then we use the same letter to denote the corresponding map $\alpha : \mathcal{C}_0 \to \mathcal{D}_1$

The term "discrete" shall refer to two different concepts. When referring to the topology on some space, then it means that all subsets are open. When referring to a category, then it means that each morphism is an identity. It shall always be clear from the context to which meaning we refer.

I 2-groups and generalised group cohomology

In this section we describe generalised cocycles and generalised cohomology in terms of 2-groups. For this we follow [Ulb81], cf. also [BL04] and [Bre92].

Definition I.1. A (unital) *2-group* is a small category $\mathcal{G}$, together with a *multiplication functor* $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, an *inversion functor* $\bar{} : \mathcal{G} \to \mathcal{G}$ and a unit object $\mathbb{1}$ (frequently identified with its identity morphism $\mathrm{id}_{\mathbb{1}}$), together with natural isomorphisms

$$\alpha_{g,k,k} : (g \otimes h) \otimes k \to g \otimes (h \otimes k)$$

for g, h, k objects of $\mathcal{G}$, called *associators*. We require $g \otimes \mathbb{1} = g = \mathbb{1} \otimes g$ and $g \otimes \overline{g} = \mathbb{1} = \overline{g} \otimes g$ on objects and morphisms and the associators are required to satisfy the coherence conditions

$$\alpha_{g,h,k \otimes l} \circ \alpha_{g \otimes h,k,l} = (\mathrm{id}_g \otimes \alpha_{h,k,l}) \circ \alpha_{g,h \otimes k,l} \circ (\alpha_{g,h,k} \otimes \mathrm{id}_l),$$

called *pentagon identity*, as well as that $\alpha_{g,h,k}$ is an identity if one of g, h or k is $\mathbb{1}$ and that $\alpha_{g,\overline{g},g} = \mathrm{id}_g$ and $\alpha_{\overline{g},g,\overline{g}} = \mathrm{id}_{\overline{g}}$.

A *morphism* of (unital) 2-groups is a functor $\mathcal{F} : \mathcal{G} \to \mathcal{G}'$, together with natural isomorphisms $\mathcal{F}_2(g, h) : \mathcal{F}(g) \otimes' \mathcal{F}(h) \to \mathcal{F}(g \otimes h)$, satisfying $\mathcal{F}(\mathbb{1}) = \mathbb{1}'$, $\mathcal{F}_2(g, \mathbb{1}) = \mathbb{1}' = \mathcal{F}_2(\mathbb{1}, g)$ and $\mathcal{F}_2(g, \overline{g}) = \mathbb{1}' = \mathcal{F}_2(\overline{g}, g)$ for all objects g of $\mathcal{G}$ and, furthermore, the coherence condition

$$\mathcal{F}_2(x, y \otimes z) \circ (\mathrm{id}_{\mathcal{F}(x)} \otimes' \mathcal{F}_2(y, z)) \circ \alpha'(\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)) =$$
$$\mathcal{F}(\alpha(x, y, z)) \circ \mathcal{F}_2(x \otimes y, z) \circ (\mathcal{F}_2(x, y) \otimes' \mathrm{id}_z).$$

A *2-morphism* between morphisms $\mathcal{F}$ and $\mathcal{F}'$ is a natural transformation $\theta : \mathcal{F} \Rightarrow \mathcal{F}'$, satisfying

$$\mathcal{F}'_2(x, y) \circ \theta(x) \otimes \theta(y) = \theta(x \otimes y) \circ \mathcal{F}_2(x, y).$$

The resulting 2-category is denoted by **2-Grp**. ∎

We took the clumsy notation for the inversion functor to distinguish it explicitly from the functor $\mathcal{G} \to \mathcal{G}^{\mathrm{op}}$ that maps each morphism to its inverse morphism.

Non-unital 2-groups involve additional natural isomorphisms, replacing the identities $g \otimes \mathbb{1} = g = \mathbb{1} \otimes g$ and $g \otimes \overline{g} = \mathbb{1} = \overline{g} \otimes g$, which are themselves required to obey certain coherence conditions. However, in our constructions these isomorphisms shall always be identities which is why we excluded them from our definition. This is also why we shall drop the adjective "unital" in the sequel.

Remark I.2. Our 2-groups are examples of coherent 2-groups, as considered in [BL04]. In particular, each morphisms is invertible in a 2-group. ∎

Example I.3. A particular important class of examples form the so called *strict* 2-groups. They can be characterised to be 2-groups, for which all natural equivalences in Definition I.1 are trivial.

Besides this description, strict 2-groups can be described by crossed modules as follows (in fact, the 2-category of strict 2-groups is 2-equivalent to the 2-category of crossed modules, cf. [Por08], [FB02]). A *crossed module* is a morphism of groups $\tau : H \to G$, together with an automorphic action of G on H, such that

$$\tau(g.h) = g \cdot \tau(h) \cdot g^{-1} \tag{1}$$
$$\tau(h).h' = h \cdot h' \cdot h^{-1}. \tag{2}$$

Note that these two equations force $\ker(\tau)$ to be central in H and $\operatorname{im}(\tau)$ to be normal in G. From this one can build up a 2-group $\mathcal{G}$ as follows. The objects are G, the morphisms are $H \rtimes G$ and the structure maps are given by $s(h,g) = g$, $t(h,g) = \tau(h) \cdot g$, $\operatorname{id}_g = (e,g)$, $(h', \tau(h) \cdot g) \circ (h,g) = (h'h, g)$. The identity object is e and the multiplication and inversion functor are given by multiplication and inversion on the groups G and $H \rtimes G$. Then (1) and (2) ensure that this defines a functor with the desired properties. In particular, the associator from Definition I.1 is trivial. We shall always refer to a strict 2-group, which is induced from a crossed module by $\mathcal{G}_{(\tau: H \to G)}$ or simply $\mathcal{G}_\tau$ if H and G are understood.

A crossed module is called *abelian* if G and H are so and G acts trivially. In this case, the associated strict 2-group is also called abelian. ■

Definition I.4. If H and G are Lie groups, then an action of G on H is called *smooth* if the map $G \times H \to H$, $(g,h) \mapsto g.h$ is smooth. With this, we define a crossed module $\tau : H \to G$ to be smooth if H and G are Lie groups and τ and the action are smooth. In this case, the associated 2-group is called a *strict Lie 2-group*. ■

Note that we only defined *strict* Lie 2-groups (for convenience via their associated crossed modules). The notion of a non-strict Lie 2-group shall be a bit more tricky, we come to this point in Section II.

Example I.5. We obtain a slightly weaker version of the previous example if we are given in addition to a crossed module $\tau : H \to G$ a group cocycle $f : G^3 \to \ker(\tau) \subseteq Z(H)$. Then we define a (non-strict) 2-group $\mathcal{G}_{\tau,\alpha}$ by almost the same assignments as $\mathcal{G}_\tau$, except that we introduce a non-trivial associator

$$\alpha(g,h,k) = (f(g,h,k), ghk).$$

Since f takes values in $\ker(\tau)$, $\alpha(g,h,k)$ is an automorphism of ghk and the cocycle condition for f is equivalent to α satisfying the pentagon identity. If, in addition, τ is trivial (and thus $H = \ker(\tau)$ is abelian), then $\mathcal{G}_{\tau,f}$ is called *skeletal*. This is the most general example of a 2-group in the sense that each other 2-group is equivalent to a skeletal one (cf. [BL04, Sect. 8.3]). ■

For the following definition, recall that $C^n(G,Z)$ denotes the group of Z-valued n-cochains on G.

Definition I.6. (cf. [Bre92, Sect. 2]) Let G be a group and $\mathcal{Z} := \mathcal{Z}_{(\tau: A \to Z)}$ be an abelian 2-group (and thus in particular strict). Then a $\mathcal{Z}$-valued generalised group 2-cocycle on G (shortly called generalised cocycle if $\mathcal{Z}$ and G are understood) consists of two maps $F \in C^2(G,Z)$ and $\Theta \in C^3(G,A)$, satisfying

$$\mathsf{d}_{\mathrm{gp}} F = \tau \circ \Theta \tag{3}$$
$$\mathsf{d}_{\mathrm{gp}} \Theta = 0. \tag{4}$$

If $\mathcal{Z}$ is, in addition, a strict Lie 2-group and G is a Lie group, then the cocycle (F, Θ) is called *smooth* if F and Θ are smooth on some unit neighbourhood.

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A morphism of generalised cocycles $(\varphi, \psi) : (F, \Theta) \to (F', \Theta')$ consists of maps $\varphi \in C^1(G, Z)$ and $\psi \in C^2(G, A)$ such that

$$F = F' + \mathsf{d}_{\mathrm{gp}}\, \varphi + \tau \circ \psi \tag{5}$$

$$\Theta = \Theta' + \mathsf{d}_{\mathrm{gp}}. \tag{6}$$

If the maps φ and ψ should define a morphism of smooth generalised cocycles, then they are required to be smooth on a unit neighbourhood.

Furthermore, a 2-morphism $\gamma : (\varphi, \psi) \Rightarrow (\varphi', \psi')$ between two morphisms of generalised cocycles is given by a map $\gamma : G \to A$, satisfying

$$\psi = \psi' + \mathsf{d}_{\mathrm{gp}}\, \gamma.$$

If (φ, ψ) and (φ', ψ') are morphisms of smooth generalised cocycles, then γ is called smooth if it is so on some unit neighbourhood of G. The resulting 2-category is denoted by $\boldsymbol{\mathcal{H}^2(G, Z)}$ (respectively $\boldsymbol{\mathcal{H}^2_s(G, Z)}$ in the smooth case). ∎

Example I.7. Let G be an arbitrary group and let $\Gamma \subseteq Z$ be a subgroup of an abelian group Z. If $F : G \times G \to Z/\Gamma$ is an ordinary group cocycle, then this gives rise to a generalised cocycle as follows. First we choose a lift $s : Z/\Gamma \to Z$, satisfying $s(0) = 0$. Since s may in general not be a homomorphism, $F^\sharp := s \circ F$ is not a cocycle. However, $\mathsf{d}_{\mathrm{gp}}\, F^\sharp$ takes values in Γ. With $\mathsf{d}_{\mathrm{gp}}^2 = 0$ it is easily checked that $(F^\sharp, \mathsf{d}_{\mathrm{gp}}\, F^\sharp)$ defines a generalised cocycle with values in $\mathcal{Z}_\tau$, where τ is the inclusion $\Gamma \hookrightarrow Z$. ∎

In this paper we shall only deal with the case that A is a discrete group. If a generalised cocycle (F, Θ) is smooth, then this implies that Θ vanishes on some identity neighbourhood, for smooth maps are in particular continuous.

Remark I.8. Generalising the construction of group cohomology in the obvious way, we see that generalised cocycles, together with their morphisms and 2-morphisms form a 2-category $\mathcal{H}^2(G, \mathcal{Z})$. Moreover, point-wise addition of mappings defines a multiplication functor, endowing $\mathcal{H}^2(G, \mathcal{Z})$ with the structure of a *strict 3-group*. ∎

Remark I.9. Put into the framework of homological algebra, the previous definition reads as follows. Interpreting A and Z as chain complexes and τ as a map between them, we obtain the cone

$$\mathrm{cone}(\tau) = \cdots \to Z \xrightarrow{\tau} A \to \cdots$$

and a short exact sequence $Z \to \mathrm{cone}(\tau) \to A[-1]$ of chain complexes. If $H^n(G, \mathrm{cone}(\tau))$ denotes the group hypercohomology of G with coefficients in the chain complex $\mathrm{cone}(\tau)$, then the calculation of $H^2(G, \mathrm{cone}(\tau))$ in terms of the bar complex shows that $H^2(G, \mathrm{cone}(\tau))$ consists of equivalence classes of generalised cocycles modulo morphisms. Moreover, the above short exact

sequence in coefficients induces a long exact sequence in group hypercohomology, which leads to an exact sequence

$$\cdots \to H^2(G, Z) \to H^2(G, \mathrm{cone}(\tau)) \to H^3(G, A) \to \cdots$$

(inducing the obvious mappings). This gives a possibility to obtain information on generalised cohomology from ordinary cohomology. For details on this point of view we refer to [Wei94].

For smooth generalised cocycles, the situation is more involved, since the category of abelian Lie groups is not abelian, for it does not possess cokernels (i.e., images are not necessarily closed). However, certain constructions may be remedied by considering categories with fewer morphisms, cf. [HS07]. ∎

Remark I.10. Note that a $\mathcal{Z}$-valued cocycle (F, Θ) yields an ordinary group 2-cocycle with values in $Z/\mathrm{im}(\tau)$, by composing F with the quotient map $q : Z \to Z/\mathrm{im}(\tau)$. We call $q \circ F$ the *band* of (F, Θ). ∎

II Smooth 2-groups

In this section we shall elaborate on our concept of smoothness of 2-groups. Since our 2-groups are internal to sets (for they are assumed to be small categories), it seems to be natural to work internal to manifolds (i.e., require sets to be manifolds and maps to be smooth), but this turns out to be too restrictive. The perspective to Lie groups that we take for our notion of Lie 2-groups is that a Lie group is a group with a locally smooth group multiplication. We make this precise in the beginning of the section.

The arguments in this section shall be valid for smooth spaces in a more general sense that just locally convex manifolds. However, to stay clear and brief we shall stick to the manifold case.

Note: When referring to a locally smooth map on a pointed manifold, then we shall always mean that the map is smooth on some neighbourhood *of the base point*. This should *not* be mixed up with the alternate usage of local smoothness, saying that a map is smooth on a neighbourhood of *each* point (which of course is equivalent to global smoothness).

Theorem II.1. *Let G be a group $U \subseteq G$ be a subset containing e such and let U be endowed with a manifold structure. Moreover, assume that there exists an open neighbourhood $V \subseteq U$ of e such that*

  i) $V^{-1} = V$ and $V \cdot V \subseteq U$,

  ii) $V \times V \ni (g, h) \mapsto gh \in U$ is smooth,

  iii) $V \ni g \mapsto g^{-1} \in V$ is smooth and

  iv) V generates G as a group.

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Then there exists a manifold structure on G such that V is open in G and such that group multiplication and inversion is smooth. Moreover, for each other choice of V, satisfying the above conditions, the resulting smooth structures on G coincide.

Proof. The proof is standard, see for instance [Bou89, Prop. III.1.9.18]. However, we shall repeat the essential parts to illustrate the general idea.

Let $W \subseteq V$ be open such that $W \cdot W \subseteq V$ and $W^{-1} = W$. Then we transport the smooth structure from W to gW by left translation $\lambda_g : W \to gW$ (i.e. we *define* λ_g to be a diffeomorphism). This is well-defined since for $gW \cap hW \neq \emptyset$ we have $h^{-1}g \in V$ so that the coordinate change

$$\lambda_g^{-1}(gW \cap hW) \ni x \mapsto \lambda_{h^{-1}g}(x) \in \lambda_h^{-1}(gW \cap hW)$$

is smooth.

To verify that the group multiplication is smooth, we first observe that from iv) it follows that for each $h \in G$ there exists a neighbourhood $W_h \subseteq V$ of e such that $h^{-1}W_h h \subseteq V$ and $x \mapsto h^{-1}xh$ is smooth (the set of all $h \in G$ such that W_h exists forms a sub monoid, containing V). Thus,

$$gW_h \times hW \ni (x, y) \mapsto xy = \lambda_{gh}\big((h^{-1} \cdot \lambda_g^{-1}(x) \cdot h) \cdot \lambda_h^{-1}(y)\big) \in V$$

is smooth. A similar argument shows that inversion is also smooth.

If V' is another open unit neighbourhood of G, satisfying the above requirements, then id_G is smooth on $V \cap V'$. Since a group homomorphism is smooth if and only if it is so on some unit neighbourhood is follows that id_G is a diffeomorphism with respect to the two smooth structures, induced from V and V' ∎

Note that the previous theorem says that a *global* group structure, which is *locally* smooth, is automatically globally smooth. For non-strict 2-groups we shall encounter a different behaviour below. The following example illustrates an application of the preceding theorem. A similar construction shall be used later for building Lie 2-groups.

Example II.2. Let G be an arbitrary connected Lie group an let $G \to C_*^\infty([0,1], G)$, $g \mapsto \alpha_g$ be a section of the map that evaluates in 1. Moreover, assume that α is smooth on some unit neighbourhood $U \subseteq G$. Then

$$\Theta_\alpha(g, h) = [\alpha_g + g.\alpha_h - \alpha_{gh}] \in H_1(G) \cong \pi_1(G)$$

is a *locally* smooth group cocycle for the universal covering group $\widetilde{G} \cong PG/(\Omega G)_0$. Thus $\widetilde{G}$ can be realised as a group by $\pi_1(G) \times_{\Theta_\alpha} G$, endowed with the multiplication

$$(a, g) \cdot (b, h) = (a + b + \Theta_\alpha(g, h), gh)$$

Now $f|_{U \times U}$ is smooth, and taking $V \subseteq U$ open with $e \in V$, $V^{-1} = V$ and $V \cdot V \subseteq U$ shows that multiplication in $\pi_1(G) \times_{\Theta_\alpha} G$ is smooth when restricted to $(V \times G)^2$ (a similar argument works for the inversion, since $(a, g)^{-1} = (-a - f(g, g^{-1}), g^{-1})$). Thus Theorem II.1 yields a smooth group structure on $\pi_1(G) \times_{\Theta_\alpha} G$, turning

$$\pi_1(G) \hookrightarrow \pi_1(G) \times_{\Theta_\alpha} G \to G$$

into the universal covering by the uniqueness part of Theorem II.1.

Note that Θ_1 is only required to be locally smooth. If it were globally smooth, then we could take $U = V = G$ is the previous construction and the above covering would possess the global section $g \mapsto (0, g)$, which is the case if and only if $\pi_1(G)$ vanishes. ∎

The following corollary is the equivalent statement to the previous theorem for *strict* 2-groups.

Corollary II.3. *Let $\mathcal{G}$ be a strict 2-group and let $U_0 \subseteq \mathcal{G}_0$, $U_1 \subseteq \mathcal{G}_1$ and $U_2 \subseteq \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ be generating subsets containing $\mathbb{1}$, $\mathrm{id}_{\mathbb{1}}$ and $(\mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}})$, respectively. Moreover, assume that there exist open neighbourhoods in U_0, U_1 and U_2 upon which all structure maps and group functors restrict to smooth maps.*

Then there exist unique smooth structures on $\mathcal{G}_0$ and $\mathcal{G}_1$, such that all structure maps and group functors are smooth.

The following proof relies heavily on the fact that strict 2-groups actually are group objects internal to the category of groups, i.e., spaces of objects, morphisms and composable morphisms are groups and all structure maps are group homomorphisms (cf. [Por08], [BL04], [FB02]).

Proof. Since the group functors endow $\mathcal{G}_0$ and $\mathcal{G}_1$ with group structures, Theorem II.1 yields smooth group structures on $\mathcal{G}_0$ and $\mathcal{G}_1$. It thus remains to show that the structure maps are smooth. But this is also immediate, for they are group homomorphisms, and thus local smoothness implies global smoothness. ∎

It might look quite promising to expect a similar construction of globally smooth 2-group structures from locally ones also in the case of non-strict 2-groups, but this expectation is too optimistic as the following constructions shows.

Remark II.4. Each 2-group comes along with a couple of natural groups associated to it.

- The set of isomorphism classes $\mathrm{Skel}(\mathcal{G})$ of $\mathcal{G}$. Since $\otimes$ is a functor, it induces a map $\mathrm{skel}(\mathcal{G}) \times \mathrm{skel}(\mathcal{G}) \to \mathrm{skel}(\mathcal{G})$, which clearly defines a group multiplication, for isomorphic objects in $\mathcal{G}$ become equal in $\mathrm{skel}(\mathcal{G})$.

- The set $\mathcal{G}_{\mathbb{1}}$ of morphisms in the full subcategory of $\mathcal{G}$, generated by $\mathbb{1}$. On $\mathcal{G}_{\mathbb{1}}$, we define a map

$$\mathcal{G}_{\mathbb{1}} \times \mathcal{G}_{\mathbb{1}} \to \mathcal{G}_{\mathbb{1}}, \quad (g, h) \mapsto g \otimes h.$$

If we assume that $\alpha_{g,h,k}$ vanishes is an identity if one of g, h or k is isomorphic to $\mathbb{1}$,[2] then this defines an associative multiplication on $\mathcal{G}_{\mathbb{1}}$. Since $f \in \mathcal{G} \Leftrightarrow \overline{f} \in \mathcal{G}_{\mathbb{1}}$, this is in fact a group.

- The source and target fibres $s^{-1}(\mathbb{1})$ and $t^{-1}(\mathbb{1})$ are a subgroup of $\mathcal{G}_{\mathbb{1}}$.

- The endomorphisms $\mathrm{End}(\mathbb{1}) = s^{-1}(\mathbb{1}) \cap t^{-1}(\mathbb{1})$ of $\mathbb{1}$ form a subgroup of $\mathcal{G}_{\mathbb{1}}$. $\blacksquare$

Lemma II.5. *Let $\mathcal{G}$ be a 2-group such that spaces of objects, morphisms and composable morphisms are manifolds, all structure maps are smooth as well as the multiplication and inversion functor and the associator. Moreover, assume that $\alpha_{g,h,k}$ is an identity if one of g, h or k is isomorphic to $\mathbb{1}$. Then then smooth structure on $\mathcal{G}_1$ endows $\mathcal{G}_{\mathbb{1}}$, $s^{-1}(\mathbb{1})$, $t^{-1}(\mathbb{1})$ and $\mathrm{End}(\mathbb{1})$ naturally with a smooth group structure and*

$$\mathcal{G}_{\mathbb{1}} \times \mathcal{G}_1 \to \mathcal{G}_1, \quad (a, f) \mapsto a \otimes f \tag{7}$$

defines a smooth action. Moreover, this action is free, $\mathcal{G}_1 / s^{-1}(\mathbb{1}) \cong \mathcal{G}_0$ as smooth spaces and $\mathcal{G}_1$ is a trivial smooth principal $s^{-1}(\mathbb{1})$-bundle.

Proof. That the action is free follows from

$$a \otimes f = b \otimes f \Rightarrow a \otimes \underbrace{(f \otimes f^{-1})}_{\in \mathrm{End}(\mathbb{1})} = b \otimes \underbrace{(f \otimes f^{-1})}_{\in \mathrm{End}(\mathbb{1})} \Rightarrow a = b.$$

The source map $\mathcal{G}_1 \to \mathcal{G}_0$ is $s^{-1}(\mathbb{1})$-invariant and thus induces a smooth map $\mathcal{G}_1 / s^{-1}(\mathbb{1}) \to \mathcal{G}_0$. The identity map $\mathcal{G}_0 \to \mathcal{G}_1$ provides a smooth global section, proving the claim. $\blacksquare$

Lemma II.6. *Let $\mathcal{G}$ be a 2-group such that spaces of objects, morphisms and composable morphisms are manifolds, all structure maps are smooth as well as the multiplication and inversion functor and the associator. If $s^{-1}(\mathbb{1})$ is discrete in the induced topology then the arrow part*

$$(\mathcal{G}_0)^3 \ni (g, h, k) \mapsto \alpha(g, h, k) \otimes \mathrm{id}_{\overline{(g \otimes h) \otimes k}} \in s^{-1}(\mathbb{1}) \subseteq \mathcal{G}_1$$

of α is locally constant.

Proof. This is due to the fact that smooth maps between locally convex manifolds are in particular continuous. $\blacksquare$

The importance of the previous lemma is that we are forced to work with 2-groups with $s^{-1}(\mathbb{1})$ discrete if we want a reasonable interpretation of a Lie 2-group integrating an ordinary Lie algebra (cf. Section IV). It illustrates the limitation on building 2-groups with too many smoothness conditions. However, locally smoothness of the group multiplication is essential for passing from Lie 2-group to Lie 2-algebras. Thus the following definition seems to be natural.

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[2]This should follow from coherence, but we were not able to find a reference for it. However, all 2-groups that we encounter in this article shall obey this condition

Definition II.7. A smooth 2-space is a small category such that all sets involved in the definition of a small category (*including* the pull-backs in the definition of composition and its associativity) are pointed smooth manifolds and all structure maps (*including* the maps defined on these pull-backs) are pointed and smooth.

A smooth functor (sometimes called smooth 1-map) between smooth 2-spaces is a functor which is pointed and smooth as a map on objects, morphisms and composable morphisms. Likewise, a smooth natural transformation (sometimes called smooth 2-map) between smooth functors is a natural transformation which is pointed and smooth as a map on objects and a smooth natural equivalence is a natural equivalence which is a smooth natural transformation and whose inverse is so. The corresponding 2-category of smooth 2-spaces is denoted by **2-Man**.

A Lie 2-group is a 2-group, such that the underlying category is a smooth 2-space and that the group functors and natural transformations are smooth when restricted to some open subcategory, containing $\mathbb{1}$. Likewise, morphisms and 2-morphisms between Lie 2-groups are morphisms and 2-morphisms of the underlying 2-group, which are smooth when restricted to some open subcategory, containing $\mathbb{1}$. The corresponding sub 2-category of **2-Man** is denoted by **Lie2-Grp**. ∎

The previous definition defines the 2-category of smooth 2-spaces to be the 2-category of internal categories in smooth manifolds. For more general purposes as we are aiming for here this definition is insufficient, for the category of smooth manifold has bad categorical properties (i.e., it lacks pull-backs, quotients and internal homs). This can be remedied by introducing smooth 2-spaces as categories internal to smooth spaces (also called Chen- or diffeological spaces), for which we refer to [BH08].

For the case of a non-strict 2-group, our notion of a Lie 2-group does *not* fit with the notion used in [BL04, Def. 27] (which is actually defined without giving examples), where the functors and natural transformation, defining the 2-group structure, are required to be globally smooth. For the reasons explained above we find our notion more natural in the non-strict case. The observation that the concept introduced in [BL04] is inadequate has also been made by Henriques in [Hen07, Sect. 9]. However, the previous definition covers strict Lie 2-groups by Corollary II.3 (cf. [BC04], [BL04] [Woc08]). Moreover, it leads to a locally smooth group structure on the skeleton (in the appropriate category where the skeleton is a smooth space) and thus a globally smooth group structure thereon by Theorem II.1. We thus may interpret our Lie 2-groups as a categorified version of a Lie group, much like Lie groupoids are categorified manifolds.

III   Integrating cocycles

This section is devoted to the integration procedure for Lie algebra cocycles to generalised smooth cocycles. This shall be done by invoking a general construction scheme for higher coverings of topological groups.

Definition III.1. A Lie algebra *cocycle* is a continuous bilinear map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ satisfying $\omega(x,y) = -\omega(y,x)$ and

$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0.$$

The cocycle ω is said to be a *coboundary* if there exists a continuous linear map $b : \mathfrak{g} \to \mathfrak{z}$ with $\omega(x,y) = b([x,y])$. The vector space of cocycles is denoted by $Z_c^2(\mathfrak{g},\mathfrak{z})$ and the space of coboundaries $B_c^2(\mathfrak{g},\mathfrak{z})$ is a sub vector space of $Z_c^2(\mathfrak{g},\mathfrak{z})$. The quotient space $H_c^2(\mathfrak{g},\mathfrak{z})$ is called the second (continuous) *Lie algebra cohomology* of $\mathfrak{g}$ with coefficients in $\mathfrak{z}$, and two cocycles ω and ω' are called *equivalent* if $[\omega] = [\omega']$ in $H_c^2(\mathfrak{g},\mathfrak{z})$. ∎

Remark III.2. Lie algebra cocycles are a concept more important for infinite-dimensional $\mathfrak{g}$ than for finite-dimensional. For instance, Whitehead's theorem asserts that $H_c^2(\mathfrak{g},\mathfrak{z})$ vanishes if $\mathfrak{g}$ is finite-dimensional and semi-simple. A prominent example for a non-trivial cocycle is the Kac–Moody cocycle

$$\omega : C^\infty(S^1, \mathfrak{k}) \times C^\infty(S^1, \mathfrak{k}) \to \mathbb{R}, \quad (f,g) \mapsto \int_{S^1} \langle f(t), g'(t) \rangle dt,$$

where $\langle \cdot, \cdot \rangle$ is the Cartan–Killing form of the finite-dimensional Lie algebra $\mathfrak{k}$. ∎

Lemma III.3. *Let $\mathcal{Z}_{(\tau:A \to Z)}$ be an abelian Lie 2-group with A discrete and $Z = \mathfrak{z}/\Gamma$. If (F,Θ) is a $\mathcal{Z}$-valued normalised and smooth generalised cocycle, then*
$$L(F) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}, (x,y) \mapsto T^2 f(x,y) - T^2 f(y,x)$$

(where $T^2 f$ is the second Taylor monomial) defines a Lie algebra cocycle.

Proof. Let $U \subseteq G$ be a unit neighbourhood such that $F|_{U \times U}$ and $\Theta|_{U \times U \times U}$ are smooth maps. Then $\Theta|_{U \times U \times U}$ is constant, since it is in particular continuous. Thus
$$F(g,h) + F(gh,k) - F(g,hk) - F(h,k) = 0$$

for g, h, k in U. Since the computation of $L(F)$ in [Nee02, Lem. 4.6] only depends on the values of F on $U \times U$, the same calculation shows the claim. ∎

Definition III.4. A $\mathcal{Z}_\tau$-valued smooth generalised cocycle (F,Θ) *integrates* a $\mathfrak{z}$-valued Lie-algebra cocycle ω if $L(F)$ is equivalent to ω. ∎

The rest of this section is devoted to the description of the integration procedure for Lie algebra cocycles and show that this procedure is universal. The following lemma describes the simplicial part of the procedure for enlarging local group cocycles to global ones (cf. [vEK64]) and the lemma afterwards encodes the data that one needs when integrating Lie algebra cocycles local group cocycles. This construction is also implicitly hidden in [BM94], at least for the transgressed Čech cocycle.

Lemma III.5. *Assume that there exist maps*

$$\alpha : G \to C_*^\infty(\Delta^{(1)}, G),$$
$$\beta : G^2 \to C_*^\infty(\Delta^{(2)}, G),$$

such that

$$\alpha_e \equiv e, \quad \alpha_g(1) = g, \quad \beta_{g,g}(s,t) = \alpha_g(s+t) \tag{8}$$
$$\partial\beta_{g,h} = \alpha_g + g.\alpha_h - \alpha_{gh}. \tag{9}$$

Then $\mathsf{d}_{\mathrm{gp}}\,\beta : G^3 \to C_2(G)$ *takes values in* $Z_2(G)$. *If we set* $\Theta_\beta := q \circ \mathsf{d}_{\mathrm{gp}}\,\beta$, *for* $q : Z_2(G) \to H_2(G)$ *the canonical quotient map, then* Θ_β *defines a group 3-cocycle with values in* $H_2(G) \cong \pi_2(G)$.

If α', β' *is another pair of maps, satisfying* (8) *and* (9), *then there exists a map* $\gamma : G \to C_*^\infty(\Delta^{(2)}, G)$ *with* $\partial\gamma_g = \alpha_g + g.c_e - \alpha'_g$, *where* c_e *is the path which is constantly* e. *Moreover,*

$$b_\gamma(g, h) = \beta_{g,h} - \beta'_{g,h} - \mathsf{d}_{\mathrm{gp}}\,\gamma \tag{10}$$

takes values in $Z_2(G)$ *and satisfies* $\Theta_\beta = \Theta_{\beta'} + q \circ \mathsf{d}_{\mathrm{gp}}\,b_\gamma$.

We took β as sole subscript, indicating the dependence of Θ on α and β, for α is completely determined by β.

Proof. From (9) it follows directly that

$$\partial(\mathsf{d}_{\mathrm{gp}}\,\beta)(g, h, k) = \partial(g.\beta_{h,k}) - \partial\beta_{gh,k} + \partial\beta_{g,hk} - \partial\beta_{g,h}$$

vanishes and thus Θ_β takes values in $Z_2(G)$. That Θ_β is a group cochain follows from $\beta_{g,g}(s,t) = \alpha_g(s+t)$, for then $\Theta_\beta(g, h, k)$ is null-homologous if one of g, h or k equals e. That Θ_β is a cocycle follows from $\mathsf{d}_{\mathrm{gp}}^2 = 0$.

Since G is simply connected, the map γ exists by [Nee02, Prop. 5.6]. Just as above it is checked that b_γ takes values in $Z_2(G)$. Moreover, $\mathsf{d}_{\mathrm{gp}}^2 = 0$ yields $q \circ \mathsf{d}_{\mathrm{gp}}\,b_\gamma = q \circ \mathsf{d}_{\mathrm{gp}}(\beta - \beta') = \Theta_\beta - \Theta_{\beta'}$. ∎

That Θ_β is a cocycle is actually trivial, since we wrote is as a coboundary of the group cochain β. The point here is that it takes values in the much smaller subgroup $Z_2(G)$ and as cocycle with values in this group it is *not* trivial. It even is the other extreme, namely *universal*, at least for discrete abelian groups, cf. [PW08]. The corresponding construction for a cocycle describing the universal covering, which is thus a universal 2-cocycle for discrete abelian groups, is given in Example II.2.

Lemma III.6. *Let* $\varphi : U \to \widetilde{U} \subseteq \mathfrak{g}$ *be a chart with* $\varphi(e) = 0$ *and* $\varphi(U)$ *convex and* $\widetilde{V} \subseteq \widetilde{U}$ *open and convex such that* $V := \varphi^{-1}(\widetilde{V})$ *contains* e *and we have* $V = V^{-1}$ *and* $V \cdot V \subseteq U$. *For* $g \in U$ *we set* $\tilde{g} := \varphi(g)$ *and define* $\tilde{g} * \tilde{h} = \widetilde{gh}$.

$$\alpha_g(t) = \varphi^{-1}(t \cdot \tilde{g}), \tag{11}$$
$$\beta_{g,h}(t, s) = \varphi^{-1}\left(t(\tilde{g} * s\tilde{h}) + s(\tilde{g} * (1-t)\tilde{h})\right) \tag{12}$$

for $g, h \in V$, then these assignments can be extended to mappings α and β satisfying (8) and (9). Moreover, for two different charts φ, ψ and $\bar{g} := \psi(g)$, the assignment

$$\gamma_g(s,t) = \varphi^{-1}\left(\frac{s(1-t)}{t+s}\varphi(\psi^{-1}((t+s)\bar{g}))) + t(1+s)\tilde{g}\right) \tag{13}$$

for $x \in V \cap V'$ can be extended to a map $\gamma : G \to C_*^\infty(\Delta^{(2)}, G)$, satisfying $\partial\gamma_g = \alpha_g - \alpha'_g$. If $W \subseteq V$ with $W \cdot W \subseteq V$, then $\Theta_\beta|_{W \times W \times W}$ and $b_\gamma|_{W \times W}$ are smooth.

Proof. It is easily checked for $g, h \in V$, that α_g and $\beta_{g,h}$ defined as above satisfy (8) and (9). Since G is connected, we may choose for each $g \in G\backslash U$ some $\alpha_g \in C^\infty(\Delta^1, G)$ with $\alpha_g(0) = e$ and $\alpha_g(1) = g$.

For $g, h \in G$ with $g \notin V$ or $h \notin V$, $\alpha_g + \lambda_g \circ \alpha_{g^{-1}h} - \alpha_h$ is in $Z_2(G)$, and thus there exists some $\beta_{g,h} \in C^\infty(\Delta^{(2)}, G)$ with $\partial\beta_{g,h} = \alpha_{e,g} + \alpha_{g,h} - \alpha_{e,h}$, for G is simply connected (cf. [Nee02, Prop. 5.6]). It is immetiate that for γ_g as defined in (13) we have $\partial\gamma_g = \alpha_g - \alpha'_g$. Since G is simply connected, we may choose for each $g \notin V \cap V'$ some γ_g with $\partial\gamma_g = \alpha_g - \alpha'_g$. The rest is immediate. ∎

We review some aspects of the integration procedure for Lie algebra cocycles to ordinary smooth group cocycles.

Remark III.7. Associated to each Lie algebra cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ is its period homomorphism $\mathrm{per}_\omega : \pi_2(G) \to \mathfrak{z}$, given on (piecewise) smooth representatives by $[\sigma] \mapsto \int_\sigma \omega^l$ (cf. [Nee02] or [Woc06a] for the fact that each homotopy class contains a smooth representative and [Nee02] for the fact $\int_\sigma \omega^l$ is independent on the choice of a representative).

We now define $F_{\omega,\beta} : G \times G \to \mathfrak{z}$ by

$$F_{\omega,\beta}(g,h) := \int_{\beta_{g,h}} \omega^l, \tag{14}$$

where $\beta : G^2 \to C_*^\infty(\Delta^{(2)}, G)$ is the map from Lemma III.6, applied to a chart φ with $d\varphi(0) = \mathrm{id}_\mathfrak{g}$, and ω^l is the left-invariant $\mathfrak{z}$-valued 2-form on G with $\omega^l(e) = \omega$. That $F_{\omega,\beta}$ is a group cochain follows, because $\beta(g,g)$ and $\beta(e,g)$ are null-homologous. In the case that $\Pi_\omega := \mathrm{per}_\omega(\pi_2(G))$ is discrete, we set $Z_\omega := \mathfrak{z}/\Pi_\omega$ and define $f : G \times G \to Z_\omega$ by $f(g,h) := [F(g,h)]$.

That f is a cocycle follows from

$$\mathrm{d}_{\mathrm{gp}} F(g,h,k) = F(h,k) - F(gh,k) + F(g,hk) - F(g,h) \tag{15}$$

$$= \int_{\beta_{h,k}} \omega^l - \int_{\beta_{gh,k}} \omega^l + \int_{\beta_{g,hk}} \omega^l - \int_{\beta_{g,h}} \omega^l \tag{16}$$

$$= \int_{g.\beta_{h,k}-\beta_{gh,k}+\beta_{g,hk}-\beta_{g,h}} \omega^l = \mathrm{per}_\omega(\Theta_\beta(g,h,k)), \tag{17}$$

for $\Theta_\beta : G \times G \times G \to H_2(G) \cong \pi_2(G)$ defined as in Lemma III.5. That F and f are smooth on a unit neighbourhood follows from the fact that the map $C^\infty(\Delta^{(2)}, G) \ni \beta \mapsto \int_{\Delta^2} \omega^l$ is smooth. ∎

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The previous construction only works if Π_ω is a discrete subgroup of $\mathfrak{z}$, for only then $\mathfrak{z}/\Pi_\omega$ is a Lie group. This is the obstruction against a Lie algebra cocycle ω to integrate to a group cocycle (cf. [Nee02, Thm. 7.9]).

We shall remedy the situation of indiscreet Π_ω in the following proof of the main theorem of the paper by considering the abelian crossed module $\mathrm{per}_\omega : \pi_2(G) \to \mathfrak{z}$ and the associated strict Lie 2-group $\mathcal{Z}_\omega$. Note that $\mathrm{skel}(\mathcal{Z}_\omega)$ is $\mathfrak{z}/\Pi_\omega$, which coincides with the Lie group Z_ω if Π_ω is discrete.

Theorem III.8. *The smooth $\mathcal{Z}_\omega$-valued generalised cocycle $(F_{\omega,\beta}, \Theta_\beta)$ from Remark III.7 integrates ω.*

Proof. First note, that (15) implies that $(F_{\omega,\beta}, \Theta_\beta)$ actually defines a generalised cocycle. It is also smooth by its construction.

We show that $(F_{\omega,\beta}, \Theta_\beta)$ actually integrates ω. Since $F|_{V \times V}$ coincides with the function $f_V : V \times V \to \mathfrak{z}$ in [Nee02, Lemma 6.2], associated to the cocycle ω and the choice of smooth maps $\sigma_{g,h} : \Delta^{(2)} \to G$, which coincide with $\beta_{g,h}$ as defined in Lemma III.6, [Nee02, Lemma 6.2] shows

$$L(F)(x, y) = T^2 f_V(x, y) - T^2 f_V(y, x) = \omega(x, y). \qquad \blacksquare$$

Remark III.9. The construction in Remark III.7 and the preceding proof depends on the actual *choice* of the map $\beta : G \times G \to C^\infty(\Delta^{(2)}, G)$, which in turn is chosen according to a chart φ. However, for two different charts φ and φ', the resulting cocycles Θ and Θ' are smoothly equivalent by Lemma III.5 and Lemma III.6. Moreover, if $\gamma : G \to C_*^\infty(\Delta^{(2)}, G)$ is the corresponding map as defined in Lemma III.6, we obtain the morphism $(\varphi, \psi) : (F_{\omega,\beta}, \Theta_\beta) \Rightarrow (F_{\omega,\beta'}, \Theta_{\beta'})$, given by $\psi(g, h) = [b_\gamma(g, h)]$ and

$$\varphi(g) = \int_{\gamma_g} \omega^l.$$

If two Lie algebra cocycles ω and ω' are equivalent, then $\omega(x, y) = \omega'(x, y) + b([x, y])$ for $b : \mathfrak{g} \to \mathfrak{z}$ linear and continuous. This leads to

$$\int_{\beta_{g,h}} (\omega - \omega')^l = \int_{\beta_{g,h}} d(b^l) = \int_{\partial \beta_{g,h}} b^l = \int_{\alpha_g} b^l + \int_{g.\alpha_h} b^l - \int_{\alpha_{gh}} b^l$$

by Stokes Theorem. We thus obtain $(\varphi, \psi) : (F_{\omega,\beta}, \Theta_\beta) \to (F_{\omega',\beta}, \Theta_\beta)$ with $\psi \equiv 0$ and

$$\varphi(g) = \int_{\alpha_g} b^l. \qquad \blacksquare$$

We conclude this section with showing that the cocycle $(F_{\omega,\beta}, \Theta_\beta)$ we constructed here is universal for generalised cocycles that integrate ω. This may be seen as a substitute for the exact sequence [Nee02, 7.12]

$$0 \to \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \xrightarrow{D} H_c^2(\mathfrak{g}, \mathfrak{z}) \xrightarrow{P} \mathrm{Hom}(\pi_2(G), Z)$$

for $Z := \mathfrak{z}/\Gamma$ with $\Gamma \subseteq \mathfrak{z}$ discrete. The next lemma is the generalisation of the injectivity of D (cf. [Nee02, Prop. 7.4]) for not necessarily discrete Γ.

Lemma III.10. *Let $F : G \times G \to \mathfrak{z}$ be a map such that*

- *F is smooth on some unit neighbourhood of G^2,*

- *$\mathsf{d}_{\mathrm{gp}} F \equiv 0$ on some unit neighbourhood of G^3,*

- *$\mathsf{d}_{\mathrm{gp}} F$ takes values in some subgroup Γ of $\mathfrak{z}$ and*

- *$L(F) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ is trivial as a Lie algebra cocycle.*

Then there exists a map $\varphi : G \to \mathfrak{z}$, smooth on some unit neighbourhood, such that $F - \mathsf{d}_{\mathrm{gp}} \varphi$ vanishes on some unit neighbourhood and takes values in Γ on $G \times G$.

Proof. First note that $L(F)$ actually defines a Lie algebra cocycle by the same argument as in Lemma III.3. Since $L(F)$ is trivial, there exists $\chi : \mathfrak{g} \to \mathfrak{z}$, continuous and linear, such that $L(F) = \chi([\cdot, \cdot])$.

Let $U, V \subseteq G$ be contractible unit neighbourhoods such that $F|_{U \times U}$ is smooth and $\mathsf{d}_{\mathrm{gp}} F|_{U \times U \times U} \equiv 0$ and $V^2 \subseteq U$. Then $(\mathfrak{z} \times G, \mathfrak{z} \times V, \mu_F, (0, e))$ with

$$\mu_F((z, g), (w, h)) := (z + w + F(g, h), gh)$$

is a local Lie group with Lie algebra $\mathfrak{z} \oplus_{L(F)} \mathfrak{g}$. Since $L(F) = \chi([\cdot, \cdot])$, we have that $\mathfrak{z} \oplus_{L(F)} \mathfrak{g} \ni (z, x) \mapsto z + \chi(x) \in \mathfrak{z}$ defines a homomorphism of Lie algebras, which we may integrate to a homomorphism of local Lie groups, given by $(z, g) \mapsto z + \varphi(g)$. That this map is a homomorphism implies that $F - \mathsf{d}_{\mathrm{gp}} \varphi$ vanishes on $V \times V$.

Since $\mathsf{d}_{\mathrm{gp}} F$ takes values in Γ, we have that $f := q \circ F : G \times G \to \mathfrak{z}/\Gamma$ (where $q : \mathfrak{z} \to \mathfrak{z}/\Gamma$ is the quotient map) is in fact an ordinary group cocycle and thus $(\mathfrak{z}/\Gamma) \times_f G$ is a group (which actually is topological, but not Hausdorff in general). Now

$$f_\varphi : V \to (\mathfrak{z}/\Gamma) \times_f G, \quad g \mapsto (q(\varphi(g)), g)$$

satisfies $f_\varphi(g) \cdot f_\varphi(h) = f_\varphi(g \cdot h)$ wherever defined. Since G is simply connected, f_φ extends[3] to a unique group homomorphism. This extension is given by $g \mapsto (\varphi'(g), g)$ for some function $\varphi' : G \to \mathfrak{z}/\Gamma$. Moreover, φ' extends $q \circ \varphi|_V$ and satisfies $f - \mathsf{d}_{\mathrm{gp}} \varphi' \equiv 0$. If we choose a lift $s : \mathfrak{z}/\Gamma \to \mathfrak{z}$ with $q(0) \mapsto 0$, then $g \mapsto s(\varphi'(g))$ for $g \notin V$ extends $\varphi|_V$ to all of G with the desired properties. ∎

The following lemma is our version of [Nee02, Thm. 7.9] for non-discrete Γ.

Lemma III.11. *Let $F : G \times G \to \mathfrak{z}$ be a map such that*

- *F is smooth on some unit neighbourhood of G^2,*

- *$\mathsf{d}_{\mathrm{gp}} F \equiv 0$ on some unit neighbourhood of G^3,*

- *$\mathsf{d}_{\mathrm{gp}} F$ takes values in some subgroup Γ of $\mathfrak{z}$ and*

---

[3]The group on the right does not need to be topological for this, cf. [HM98, Cor. A.2.26]

- $L(F)$ *is equivalent to ω as a Lie algebra cocycle.*

Then $\mathrm{per}_\omega(\pi_2(G)) \subseteq \Gamma$.

Proof. Since per_ω does only depend on the cohomology class of ω (cf. [Nee02, Rem. 5.9]), we may assume that $L(F) = \omega$. Set $\Theta := \mathsf{d}_{\mathrm{gp}} F$ and let $U, V \subseteq G$ be an open and contractible unit neighbourhoods such that $F|_{U \times U}$ is smooth, $\Theta|_{U \times U \times U}$ vanishes and $V \cdot V \subseteq U$. For each $g \in G$ we define $\kappa_g \in \Omega^1(gV, \mathfrak{z})$ by

$$\kappa_g(w_x) = d_2 F_{g^{-1} \cdot x}(x^{-1}.w_x) \text{ for } w_x \in T_x\, gV,$$

where $d_2 F_g(w_h) := dF(0_g, w_h)$ for $g, h \in U$. This is smooth for $F|_{U \times U}$ is smooth and a straight forward computation shows $d\kappa_g = \omega^l\big|_{gV}$. For $g, h \in G$ with $gV \cap hV \neq \emptyset$ we have $g^{-1}h \in U$. Thus $\mathsf{d}_{\mathrm{gp}} F(g^{-1}h, h^{-1}x, x^{-1}\eta(t))$ vanishes for $\eta(t) \in gV \cap hV$, which implies

$$(\kappa_g - \kappa_h)(w_x) = d_2 F_{g^{-1} \cdot h}(h^{-1}.w_x).$$

If $\alpha : [0,1] \to gV \cap hV$ is smooth, then this in turn yields

$$
\begin{aligned}
\int_\alpha \kappa_g - \kappa_h &= \int_0^1 d_2 F_{g^{-1}h}(h^{-1}.\dot\alpha(t))dt \\
&= F(g^{-1}h, h^{-1}\alpha(1)) - F(g^{-1}h, h^{-1}\alpha(0)) \\
&= F(h, h^{-1}\alpha(1)) - F(g, g^{-1}\alpha(1)) + \Theta(g, g^{-1}h, h^{-1}\alpha(1)) - \\
&\quad F(h, h^{-1}\alpha(0)) + F(g, g^{-1}\alpha(0)) - \Theta(g, g^{-1}h, h^{-1}\alpha(0)).
\end{aligned}
$$

Now let $[\sigma] \in \pi_2(G)$ be represented by a smooth map $\sigma : [0,1]^2 \to G$, such that σ vanishes on some neighbourhood of the boundary of $[0,1]^2$. Then there exists some $n \in \mathbb{N}$ and for $i, j \in \{0, ..., n\}$ some $g_{ij} \in G$ such that

$$\sigma\left(\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]\right) \subseteq g_{ij}V.$$

We denote by σ_{ij} the restriction of σ to $[\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}]$. Then

$$\mathrm{per}_\omega([\sigma]) = \int_\sigma \omega^l = \sum_{i,j=1}^n \int_{\sigma_{ij}} \omega^l = \sum_{i,j=1}^n \int_{\sigma_{ij}} d\kappa_{g_{ij}} = \sum_{i,j=1}^n \int_{\partial \sigma_{ij}} \kappa_{g_{ij}} \qquad (18)$$

by Stokes Theorem. We parametrise the intersection $\sigma_{ij} \cap \sigma_{i+1\,j}$ by $\mu_{ij}(t) := \sigma(\frac{i+1}{n}, \frac{j+t}{n})$ and $\sigma_{ij} \cap \sigma_{ij+1}$ by $\nu_{ij}(t) = \sigma(\frac{i+t}{n}, \frac{j+1}{n})$. In particular, we have the identities

$$
\begin{array}{lll}
\mu_{i0}(0) = \mu_{n-1\,j}(1) = e & \mu_{ij}(1) = \nu_{ij}(1) & \mu_{ij}(1) = \nu_{i+1\,j}(0) \\
\nu_{0j}(0) = \nu_{in-1}(1) = e & \mu_{ij+1}(0) = \nu_{ij+1}(1) & \mu_{ij+1}(0) = \nu_{i+1\,j}(0).
\end{array}
$$

Since $\sigma|_{\partial[0,1]^2}$ vanishes, the integrals along $\partial\sigma_{ij} \cap \partial\sigma$ in (18) vanish and modulo a sum of values of Θ, we thus have

$$\mathrm{per}_\omega([\sigma]) = \sum_{i,j=0}^{n-1} \int_{\mu_{ij}} \kappa_{g_{ij}} - \kappa_{g_{i+1\,j}} - \sum_{i,j=0}^{n-1} \int_{\nu_{ij}} \kappa_{g_{ij}} - \kappa_{g_{i\,j+1}} =$$

$$\sum_{i,j=0}^{n-1} F(g_{i+1\,j}, g_{i+1\,j}^{-1}\mu_{ij}(1)) - F(g_{ij}, g_{ij}^{-1}\mu_{ij}(1)) +$$

$$\sum_{i,j=0}^{n-1} -F(g_{i+1\,j}, g_{i+1\,j}^{-1}\mu_{ij}(0)) + F(g_{ij}, g_{ij}^{-1}\mu_{ij}(0)) +$$

$$\sum_{i,j=0}^{n-1} -F(g_{i\,j+1}, g_{i\,j+1}^{-1}\nu_{ij}(1)) + F(g_{ij}, g_{ij}^{-1}\nu_{ij}(1)) +$$

$$\sum_{i,j=0}^{n-1} F(g_{i\,j+1}, g_{i\,j+1}^{-1}\nu_{ij}(0)) - F(g_{ij}, g_{ij}^{-1}\nu_{ij}(0)).$$

From the above identities it follows that the correspondingly underlined terms cancel out and the right hand side vanishes. Thus $\mathrm{per}_\omega([\sigma])$ is contained in $\langle \mathrm{im}(\Theta) \rangle \subseteq \Gamma$. ∎

Proposition III.12. *Let $\mathcal{Z}_{(\tau:A\to Z)}$ be an abelian Lie 2-group with discrete A and $Z = \mathfrak{z}/\Gamma$ for $\Gamma \subseteq \mathfrak{z}$ discrete. If (F', Θ') is a $\mathcal{Z}_\tau$-valued cocycle on G that integrates ω, then there exists a morphisms $p : \pi_2(G) \to A$ such that*

$$
\begin{array}{ccc}
\pi_2(G) & \xrightarrow{\ p\ } & A \\
\Big\downarrow{\scriptstyle\mathrm{per}_\omega} & & \Big\downarrow{\scriptstyle\tau} \\
\mathfrak{z} & \xrightarrow{\ q\ } & Z
\end{array}
$$

commutes, and there exists a morphism

$$(\varphi, \psi) : (q \circ F_{\omega,\beta}, p \circ \Theta_\beta) \to (F', \Theta')$$

of smooth generalised cocycles. Moreover, if $F' : G \times G \to Z$ is already a group cocycle, then we may assume that $\tau \circ p = q \circ \mathrm{per}_\omega$ and ψ vanish.

Proof. We abbreviate $\Theta := \Theta_\beta$ and $F := F_{\omega,\beta}$. From [PW08] we recall that Θ is universal, i.e., for each discrete abelian group B and each group 3-cocycle $\vartheta : G \times G \times G \to B$, vanishing on some unit neighbourhood, we have a unique homomorphism $p' : \pi_2(G) \to B$ satisfying $p' \circ \Theta = \vartheta$. This also implies that $\mathrm{im}(\Theta)$ generates $\pi_2(G)$.

Since $q : \mathfrak{z} \to Z$ is a covering map there exists a section $s : Z \to \mathfrak{z}$ such that $s(0) = 0$ and s is smooth on some zero neighbourhood. We set $F^\sharp := s \circ F'$ (cf. Example I.7).

We shall first consider that case where F' is already a group cocycle or, equivalently, $\mathsf{d}_{\mathrm{gp}} F^\sharp$ takes values in Γ. Then Θ' takes values in $\ker(\tau)$ and the universality of Θ yields a homomorphism $p' : \pi_2(G) \to \ker(\tau)$ satisfying $p' \circ \Theta = \Theta'$. Moreover, Lemma III.11 implies $\mathrm{per}_\omega(\pi_2(G)) \subseteq \Gamma$. Thus $q \circ \mathrm{per}_\omega = 0 = \tau \circ p'$ and we may set $p := p'$. Since $L(q \circ F) = L(F')$ is equivalent to ω by assumption, we obtain from Lemma III.10 that

$$q \circ F - F' = \mathsf{d}_{\mathrm{gp}} \varphi$$

for $\varphi : G \to Z$, which is smooth on some unit neighbourhood. Setting ψ to 0 finishes the construction in this case.

Now assume that F' does not satisfy $\mathsf{d}_{\mathrm{gp}} F' \equiv 0$. Then Lemma III.11 implies $\mathrm{per}_\omega(\pi_2(G)) \subseteq q^{-1}(\tau(A))$ and Lemma III.10 yields a map $\varphi : G \to Z$, smooth on a unit neighbourhood, such that $q \circ F - F' - \mathsf{d}_{\mathrm{gp}} \varphi$ vanishes on some unit neighbourhood and takes values in $\tau(A)$ on all of $G \times G$. Thus there exists a map $\psi : G \times G \to A$, vanishing on some unit neighbourhood, satisfying (5), i.e.,

$$q \circ F - F' - \mathsf{d}_{\mathrm{gp}} \varphi = \tau \circ \psi. \tag{19}$$

Now $\mathsf{d}_{\mathrm{gp}} \psi$ is an A-valued cocycle and thus there exists a homomorphism $p'' : \pi_2(G) \to A$ satisfying $p'' \circ \Theta = \mathsf{d}_{\mathrm{gp}} \psi$. With this we set $p := p' + p''$ for p' constructed as above. Since p' satisfies $p' \circ \Theta = \Theta'$, this yields immediately $p \circ \Theta - \Theta' = \mathsf{d}_{\mathrm{gp}} \psi$. Moreover, with (19) one verifies

$$\tau \circ p \circ \Theta = \tau \circ (p' \circ \Theta + p'' \circ \Theta) = \tau \circ (\Theta' + \mathsf{d}_{\mathrm{gp}} \psi) =$$
$$(\mathsf{d}_{\mathrm{gp}} F') + \mathsf{d}_{\mathrm{gp}}(\tau \circ \psi) = \mathsf{d}_{\mathrm{gp}}(q \circ F) = q \circ \mathrm{per}_\omega \circ \Theta$$

and thus $q \circ \mathrm{per}_\omega = \tau \circ p$ for $\mathrm{im}(\Theta)$ generates $\pi_2(G)$. ∎

IV Generalised central extensions

In this section we define central extensions and show how they arise from generalised cocycles. Our setting shall be tailored to treat central extensions that come from group cocycles, for the more general setting see [Bre92].

In the first part of the section we shall describe the route from generalised cocycles to generalised central extensions, but not the way back. The general setup of this Schreier-like theory has been worked out in [Bre92]. Moreover, we assume the familiarity of central extensions and their description by cocycles (cf. [Rot95, Ch. 7], [Wei94, Sect. 6.6] or [ML63, Sect. IV.3] for textbook introductions for ordinary groups, [Nee02] for topological and Lie groups and [AM08] for generalisations for ordinary groups). The approach to Schreier-like invariants for extensions of groupoids in [BBF05] does not fit into our situation, for our sequences of groupoids shall not be bijectively on objects.

The second part of this section elaborated on the basic notions of Lie theory for Lie 2-groups and central extensions.

Definition IV.1. For a group G, denote by $\underline{G}$ the 2-group with $\mathrm{Obj}(\underline{G}) = G$, only identity morphisms and the canonical 2-group structure. If $\mathcal{Z}$ is an abelian 2-group, then a generalised central extension of $\underline{G}$ by $\mathcal{Z}$ is a short exact sequence of 2-groups

$$\mathcal{Z} \xrightarrow{\iota} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}, \tag{20}$$

with $\iota(a) \otimes g = g \otimes \iota(a)$ for all morphisms a of $\mathcal{Z}$ and g of $\widehat{\mathcal{G}}$. Here, exactness

- in $\mathcal{Z}$ means that ι is injective on morphisms,

- in $\widehat{\mathcal{G}}$ means $(g \in \mathrm{im}(\iota_1)) \Leftrightarrow (q_1(g) = \mathbb{1})$ for each morphisms g of $\widehat{\mathcal{G}}$,

- in $\underline{G}$ means that q is surjective on morphisms.

If $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}'$ are generalised central extensions of G by $\mathcal{Z}$, then a morphism of generalised central extensions is a morphism of 2-groups $\Xi : \widehat{\mathcal{G}} \to \widehat{\mathcal{G}}'$, such that

$$
\begin{array}{ccccc}
\mathcal{Z} & \xrightarrow{\iota} & \widehat{\mathcal{G}} & \xrightarrow{q} & \underline{G} \\
\| & & \Xi \downarrow & & \| \\
\mathcal{Z} & \xrightarrow{\iota'} & \widehat{\mathcal{G}}' & \xrightarrow{q'} & \underline{G}
\end{array} \tag{21}
$$

commutes (as an equality of functors *on the nose*). A 2-morphism between morphisms of generalised central extensions is a 2-morphism between morphisms of 2-groups. ∎

Note that we may assume that $\mathcal{Z}$ is actually embedded into $\widehat{\mathcal{G}}$ as a subcategory. Since we do not want to alter the way how $\mathcal{Z}$ embeds into a generalised central extension, constructed from a generalised cocycle (as below), and for $\underline{G}$ has only identity morphisms, we require (21) to commute on the nose.

We now describe how to obtain generalised central extensions from generalised group cocycles.

Remark IV.2. Let $\mathcal{Z} = \mathcal{Z}_{(\tau : A \to Z)}$ be an abelian 2-group and (F, Θ) be a normalised generalised $\mathcal{Z}$-valued cocycle, given by cochains

$$
\begin{aligned}
\Theta &: G \times G \times G \to A \\
F &: G \times G \to Z,
\end{aligned}
$$

satisfying (3) and (4). Then the following assignment defines a normalised 2-group $\widehat{\mathcal{G}} := \widehat{\mathcal{G}}_{(F,\Theta)}$. The category $\widehat{\mathcal{G}}$ is given by

$$
\begin{aligned}
&\widehat{\mathcal{G}}_0 = Z \times G && s(a, x, g) = (x, g), \ t(a, x, g) = (\tau(a) + x, g) \\
&\widehat{\mathcal{G}}_1 = A \times Z \times G && \mathrm{id}_{(x,g)} = (0, x, g), \ (a, x, g) \circ (b, x, g) = (a + b, x, g)
\end{aligned}
$$

and the multiplication functor by

$$(a, x, g) \otimes (b, y, h) = (a + b, x + y + F(g, h), gh).$$

Since F satisfies the cocycle identity only only up to correction by Θ, this assignment defines a monoidal category if we set the associator to

$$\alpha_{(x,g),(y,h),(z,k)} = (\Theta(g,h,k), x+y+z+F(g,h)+F(gh,k), ghk)$$

and the unit object to $\mathbb{1} = (0,e)$. We clearly have $1 \otimes g = g = g \otimes \mathbb{1}$, the source-target matching condition of α is equivalent to $\mathsf{d}_{\mathrm{gp}} F = \tau \circ \Theta$ and the pentagon identity is equivalent to $\mathsf{d}_{\mathrm{gp}} \Theta = 0$. Moreover,

$$\overline{(a,x,g)} = (-a, -x - F(g,g^{-1}), g^{-1})$$

defines an inversion functor on $\widehat{\mathcal{G}}$, turning it into a 2-group.

There are canonical injections $\mathcal{Z} \hookrightarrow \widehat{\mathcal{G}}$ and surjections $\widehat{\mathcal{G}} \twoheadrightarrow G$, turning

$$\mathcal{Z} \hookrightarrow \widehat{\mathcal{G}} \twoheadrightarrow G$$

into a generalised central extension. If $(\varphi, \psi) : (F, \Theta) \to (F', \Theta')$ is a morphism of generalised cocycles, then

$$
\begin{aligned}
\mathcal{F}_0(x,g) &= (x + \varphi(g), g) \\
\mathcal{F}_1(a,x,g) &= (a, x + \varphi(g), g) \\
\mathcal{F}_2((x,g),(y,h)) &= (\psi(g,h), x + \varphi(g) + y + \varphi(h) + F'(g,h), gh)
\end{aligned}
$$

defines a morphism $\mathcal{F}_{(\varphi,\psi)}$ of normalised 2-groups and, moreover, of generalised central extensions. Likewise, for a 2-morphism $\gamma : (\varphi, \psi) \Rightarrow (\varphi', \psi')$, the assignment $(x,g) \mapsto (\gamma(g), x + \varphi(g), g)$ defines a 2-morphism between $\mathcal{F}_{(\varphi,\psi)}$ and $\mathcal{F}_{(\varphi,\psi)}$. $\blacksquare$

We now turn to generalised central extensions in the smooth setting.

Definition IV.3. Let G be an arbitrary Lie group and $\mathcal{Z} := \mathcal{Z}_{(\tau:A\to Z)}$ be a strict abelian Lie 2-group with A discrete. Then a *smooth generalised central extension* (s.g.c.e.), is a generalised central extension

$$\mathcal{Z} \xrightarrow{\iota} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}$$

of Lie 2-groups such there exists a smooth functor $q : \underline{U} \to \widehat{\mathcal{G}}$ satisfying $q \circ s = \mathrm{id}_{\underline{U}}$, where $U \subseteq G$ is some unit neighbourhood. A morphism of smooth generalised extensions $\Xi : \widehat{\mathcal{G}} \to \widehat{\mathcal{G}}'$ is a morphism of generalised central extensions, which is also a morphism of Lie 2-groups (and respectively for 2-morphisms). $\blacksquare$

The requirement on A to be discrete shall enable us to take an easy way back to Lie algebras from smooth generalised central extensions (cf. Proposition IV.12). Is is not crucial for the definition to make sense.

The following lemma is immediate from the definitions.

Lemma IV.4. *Let G be a Lie group and $\mathcal{Z}_{(\tau:A\to Z)}$ be a strict abelian Lie 2-group with A discrete. If (F,Θ) is a smooth $\mathcal{Z}$-valued generalised cocycle on G, then the 2-group $\widehat{\mathcal{G}}_{(F,\Theta)}$ from IV.2 is canonically a Lie 2-group and we have a s.g.c.e.*

$$\mathcal{Z} \hookrightarrow \widehat{\mathcal{G}}_{(F,\Theta)} \to \underline{G}. \tag{22}$$

Moreover, if $(\varphi,\psi):(F,\Theta)\to(F',\Theta')$ is a smooth morphism, then this induces a morphism $\widehat{\mathcal{G}}_{(F,\Theta)} \to \widehat{\mathcal{G}}_{(F',\Theta')}$ of s.g.c.e.

The following proposition describes the way back from generalised central extensions to ordinary ones. It is the categorical version of the discreteness condition for $\mathrm{per}_\omega(\pi_2(G))$ from [Nee02].

Proposition IV.5. *Let $\mathcal{Z} := \mathcal{Z}_{(\tau:A\to Z)}$ be a strict abelian Lie 2-group,*

$$\mathcal{Z} \xrightarrow{\iota} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G} \tag{23}$$

be a s.g.c.e. such that $\tau(A) \subseteq Z$ is a split Lie subgroup. Moreover, let $\mathrm{skel}(\mathcal{Z})$ and $\mathrm{skel}(\widehat{\mathcal{G}})$ denote the groups of isomorphism classes of $\mathcal{Z}$ and $\widehat{\mathcal{G}}$ and $\mathrm{skel}(\iota)$ and $\mathrm{skel}(q)$ denote the induced homomorphisms. Then $\mathrm{skel}(\mathcal{Z})$ and $\mathrm{skel}(\widehat{\mathcal{G}})$ carry natural Lie group structures, turning

$$\mathrm{skel}(\mathcal{Z}) \xrightarrow{\mathrm{skel}(\iota)} \mathrm{skel}(\widehat{\mathcal{G}}) \xrightarrow{\mathrm{skel}(q)} G \tag{24}$$

into a central extension of Lie groups.

Proof. First we note that $Z' := \mathrm{skel}(\mathcal{Z}) \cong Z/\tau(A)$ has a natural Lie group structure for $\tau(A)$ is split. Let $s : \underline{U} \to \widehat{\mathcal{G}}$ be a smooth section of q. Then $\mathrm{skel}(q)^{-1}(U) \cong Z' \times U$ as a set and we endow $\mathrm{skel}(q)^{-1}(U)$ with the smooth structure making this identification a diffeomorphism. Since the group multiplication on $\widehat{\mathcal{G}}$ is smooth on an open subcategory, containing $\mathbb{1}$, there exists an unit neighbourhood in $\mathrm{skel}(q)^{-1}(U)$ on which the group multiplication in $\mathrm{skel}(\widehat{\mathcal{G}})$ restricts to a smooth map. Since U generates G, $\mathrm{skel}(q)^{-1}(U)$ generates $\mathrm{skel}(\widehat{\mathcal{G}})$ and the assertion follows from Theorem II.1. ∎

Definition IV.6. The induced central extension (24) shall be called the *band* of the s.g.c.e. (23). ∎

Corollary IV.7. *Let $\mathcal{Z} := \mathcal{Z}_{(\tau:A\to\mathfrak{z})}$ be a strict abelian Lie 2-group with $\tau(A)$ discrete. If $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ is a Lie algebra cocycle and (F,Θ) is a smooth generalised $\mathcal{Z}$-valued cocycle, integrating ω, then the band of $\widehat{\mathcal{G}}_{(F,\Theta)}$ is a principal $\mathfrak{z}/\tau(A)$-bundle over G, which is a central extension, integrating $\mathfrak{z} \to \mathfrak{z} \oplus_\omega \mathfrak{g} \to \mathfrak{g}$.*

Proof. To see that the band of $\widehat{\mathcal{G}}_{(F,\Theta)}$ integrates $\mathfrak{z} \to \mathfrak{z} \oplus_\omega \mathfrak{g} \to \mathfrak{g}$ we first observe that $Tq(e) : \mathfrak{z} = T_e Z \to T_e(Z/\tau(A))$ is an isomorphism for $\tau(A)$ is discrete. Using this isomorphism to identify $\mathfrak{z}$ with $T_e(\mathfrak{z}/\tau(A))$ the claim now follows from $D(q \circ F) = Tq(e) \circ L(F)$. ∎

We thus get from s.g.c.e. to central extensions of Lie algebras in the case that $\tau(A) \subseteq Z$ is a split Lie subgroup by first taking the band of the s.g.c.e. and then passing to the derived central extension. One can also go directly from s.g.c.e. to central extensions of Lie algebras, which we shall describe now. The crucial point of this passage is that it allows one to talk of a Lie 2-group, integrating a Lie algebra.

Remark IV.8. Similarly to the concept of a smooth 2-space (and smooth functors and natural transformations, cf. Definition II.7), one defines (topological) *vector 2-spaces* to be internal categories in locally convex vector spaces, i.e., small categories, such that all sets occurring in the definition of a small category are locally convex spaces and all structure maps are continuous linear maps. Likewise, linear functors and natural transformations are defined internally, defining the 2-category **2-Vect**.

There is a natural functor T from the category **Man$_{\mathbf{pt}}$** (of pointed manifolds with smooth base-point preserving maps) to the category **Vect** (of topological vector spaces with continuous linear maps), sending manifolds to the tangent spaces at the base-point and smooth maps to their differentials at the base-point. Since this functor preserves pull-backs, it maps categories, functors an natural transformation in **Man$_{\mathbf{pt}}$** to ones in **Vect** and thus defines a 2-functor $\mathcal{T} : \textbf{2-Man} \to \textbf{2-Vect}$. ∎

If we want to enforce $\mathcal{T}$ to take values in **Vect** instead of **2-Vect**, then we need a canonical identification of $\mathcal{T}(\mathcal{M})_0$ and $\mathcal{T}(\mathcal{M})_1$. This is the case if $\mathcal{M}$ is étale, as defined below.

Definition IV.9. A smooth 2-space is called *étale* if all structure maps are local diffeomorphisms. A Lie 2-group is called étale if its structure maps are local diffeomorphisms, when restricted to some open subcategory, containing $\mathbb{1}$. Morphisms and 2-morphisms for étale Lie 2-groups are defined to be morphisms and 2-morphisms of Lie 2-groups. The corresponding 2-category is denoted by **Lie2-Grp$_{\mathbf{ét}}$**. ∎

Most of the Lie 2-groups that we shall encounter in this article are étale. Note that in the differentials of local diffeomorphisms give canonical identifications of the tangent spaces at the base-points. Thus $\mathcal{T}(\mathcal{M})$ is in fact a vector space for $\mathcal{M}$ étale. We shall make this precise for Lie 2-groups below. Note also that $s^{-1}(\mathbb{1})$ is discrete in an étale Lie 2-group. In particular, Lemma II.6 applies to étale Lie 2-groups with globally smooth group operations.

Lemma IV.10. *If $\mathcal{Z} \xrightarrow{\iota} \widehat{\mathcal{G}} \to \underline{G}$ is a s.g.c.e., then $\widehat{\mathcal{G}}$ is an étale Lie 2-group.*

Proof. It follows directly from the definition of a smooth generalised central extension that $\widehat{\mathcal{G}}\Big|_{q^{-1}(U)}$ is an open subcategory, smoothly isomorphic to $\mathcal{Z} \times \underline{U}$. ∎

Remark IV.11. Let $\mathcal{G}$ be an étale Lie 2-group. Then $\mathcal{T}(\mathcal{G})$ is a 2-vector space and the identity map gives a canonical identification of $\mathcal{T}(\mathcal{G})_0$ with $\mathcal{T}(\mathcal{G})_1$ and

all structure maps are the identity with respect to this identification. Thus it is the same as an ordinary vector space, which we denote by $\mathcal{L}(\mathcal{G})$.

The multiplication functor is given on objects by a map $m_0 : \mathcal{G}_0 \times \mathcal{G}_0 \to \mathcal{G}_0$, which is smooth on a neighbourhood of $\mathbb{1}$. Now let β be the bilinear form $T^2 m_0 : \mathcal{L}(\mathcal{G}) \times \mathcal{L}(\mathcal{G}) \to \mathcal{L}(\mathcal{G})$ and let $b(x, y) := \frac{1}{2}\beta((x, y), (x, y))$ be the corresponding quadratic form. Moreover, set

$$[x, y] := b_{\text{as}}(x, y) := (b(x, y) - b(y, x)).$$

Since $g \mapsto m_0(g, \mathbb{1})$ and $g \mapsto m_0(\mathbb{1}, g)$ is the identity map on $\mathcal{G}_0$, it has no Taylor monomials of order ≥ 2. Hence $\beta_i(0, x) = 0 = \beta_i(0, y)$, implying

$$b_i(x, y) = \beta_i((x, 0), (0, y)).$$

Thus $[\cdot, \cdot]$ is bilinear, continuous and skew-symmetric.

Let $\alpha : (\mathcal{G}_0)^3 \to \mathcal{G}_1$ be the associator in $\mathcal{G}$. We shall show that α is trivial on some unit neighbourhood. Since $\mathcal{G}$ is étale, the identity map $\mathcal{G}_0 \to \mathcal{G}_1$ is a local inverse around $\mathbb{1}$ for both, s and t. Since $\alpha_{\mathbb{1},\mathbb{1},\mathbb{1}} = \mathbb{1}$, we thus have $\text{id} \circ t \circ \alpha = \text{id} \circ s \circ \alpha$, which implies $s \circ \alpha = t \circ \alpha$ on some neighbourhood of $\mathbb{1}$. Now multiplying $\alpha(g, h, k)$ with $\text{id}_{\overline{m(m(g,h),k)}}$ defines a map with values in $s^{-1}(\mathbb{1})$, which is continuous on some unit neighbourhood and thus constantly $\mathbb{1}$. Since $\alpha_{x,\overline{x},x}$ is an identity for each x and α is natural, all of this implies

$$\text{id}_{m(m(g,h),k)} = (\alpha_{g,h,k} \otimes \text{id}_{\overline{m(m(g,h),k)}}) \otimes \text{id}_{m(m(g,h),k)} = \alpha_{g,h,k}$$

on some unit neighbourhood. This implies

$$m_0 \circ (\text{id}_{\mathcal{G}_0} \times m_0) = m_0 \circ (m_0 \times \text{id}_{\mathcal{G}_0})$$

on some unit neighbourhood, and consequently

$$b(x, b(y, z)) = b(b(x, y), z).$$

This yields the Jacobi identity for $[\cdot, \cdot]$ and thus $(\mathcal{L}(\mathcal{G}), [\cdot, \cdot])$ is a Lie algebra.

A similar argument as above shows that a morphism $\mathcal{F} : \mathcal{G} \to \mathcal{G}'$ of Lie 2-groups, where $\mathcal{G}$ and $\mathcal{G}'$ are étale, satisfies

$$m' \circ (\mathcal{F} \times \mathcal{F}) = \mathcal{F} \circ m$$

on some unit neighbourhood, and thus induces a morphism of Lie 2-algebras $\mathcal{L}(\mathcal{F}) : \mathcal{L}(\mathcal{G}) \to \mathcal{L}(\mathcal{G})'$. Likewise, a 2-morphisms θ between two such morphisms has to be the identity on some unit neighbourhood, so that $\mathcal{L}(\theta)$ is the identity natural transformation. Summarising,

$$\mathcal{L} : \textbf{Lie2-Grp}_{\textbf{ét}} \to \textbf{LieAlg}$$

defines a 2-functor from the full sub-2-category of étale Lie 2-groups to the category of Lie algebras, considered as a 2-category with only identity 2-morphisms.∎

Proposition IV.12. *Let* $\mathcal{Z} \xrightarrow{\iota} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}$ *be a s.g.c.e. Then*

$$\mathcal{L}(\mathcal{Z}) \xrightarrow{\mathcal{L}(\iota)} \mathcal{L}(\widehat{\mathcal{G}}) \xrightarrow{\mathcal{L}(q)} \mathcal{L}(\underline{G}) \qquad (25)$$

is a central extension of Lie algebras.

Proof. That (25) is exact follows from the exactness of $\mathcal{Z} \xrightarrow{\iota} \widehat{\mathcal{G}} \xrightarrow{q} G$ on objects and the differential of a section (on objects) of $\mathcal{Z} \xrightarrow{\iota} \widehat{\mathcal{G}} \xrightarrow{q} G$ provides a section of (25). ∎

Definition IV.13. For a s.g.c.e. $\mathcal{Z} \xrightarrow{\iota} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}$, its *derived* central extension is the central extension (25).

If $\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a central extension of discrete Lie 2-algebras, then this central extension is said to *integrate* to a generalised central extension if there exists a s.g.c.e. such that the derived central extension is equivalent to $\widehat{\mathfrak{g}}$. ∎

For the next theorem recall that we always assume that $\mathfrak{g}$ is the Lie algebras of a simply connected Lie group G.

Theorem IV.14. *Each central extension* $\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$ *of Lie algebras, integrates to a smooth generalised central extension of Lie 2-groups.*

Proof. We may assume that $\widehat{\mathfrak{g}}$ is equivalently given by a Lie algebra cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$, which we integrate to a $\mathcal{Z}_{\mathrm{per}_\omega}$-valued cocycle $(F_{\omega,\beta}, \Theta_\beta)$ by Theorem III.8 for some appropriate choice of β. Then Lemma IV.4 yields a s.g.c.e. $\mathcal{Z}_{\mathrm{per}_\omega} \to \widehat{\mathcal{G}}_{(F_{\omega,\beta}, \Theta_\beta)} \to \underline{G}$.

Let $U, V \subseteq G$ be open unit neighbourhood such that $F|_{U \times U}$ and $\Theta|_{U \times U \times U}$ are smooth and $V \cdot V \subseteq U$. To calculate the derived central extension, we consider the restriction of the multiplication functor m to the full subcategory with objects in $\mathfrak{z} \times U$, where it is given by

$$m_0((z,g),(w,h)) = (z + w + F_{\omega,\beta}(g,h), gh)$$

on objects. Since the second order Taylor monomial of $(z,w) \mapsto z + w$ vanishes, the second order Taylor monomial of m_0 is given by

$$((z,x),(w,y)) \mapsto (L(F)(x,y), [x,y])$$

and $L(F) = \omega$ shows the claim. ∎

We thus recover the classical case of central extensions by passing from a generalised central extension to its band in the case that $\mathrm{per}_\omega(\pi_2(G)) \subseteq \mathfrak{z}$ is discrete.

We conclude this paper with the following generalisation of Lie's Third Theorem. We give the definition of a locally exponential Lie algebra in the Appendix, one big class of them are Banach–Lie algebras.

Theorem IV.15. *Let* $\mathfrak{g}$ *be a locally exponential Lie algebra. Then there exists an étale Lie 2-group* $\mathcal{G}$, *such that* $\mathcal{L}(\mathcal{G})$ *is isomorphic to* $\mathfrak{g}$.

Proof. We consider the central extension

$$\mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_{\mathrm{ad}} := \mathfrak{g}/\mathfrak{z}(\mathfrak{g}). \qquad (26)$$

of Lie algebras. Then $\mathfrak{g}_{\mathrm{ad}}$ integrates to a locally exponential Lie group G_{ad}, which we may assume to be simply connected (cf. [Nee06, Th. IV.3.8]). This integrates by Theorem IV.14 to a smooth generalised central extension $\mathcal{Z} \to \mathcal{G} \to G_{\mathrm{ad}}$. In particular, $\mathcal{G}$ is étale. Since $\mathcal{L}(\mathcal{Z}) \to \mathcal{L}(\mathcal{G}) \to \mathcal{L}(G_{\mathrm{ad}})$ is equivalent to (26), $\mathcal{L}(\mathcal{G})$ is in particular isomorphic to $\mathfrak{g}$. ∎

V Prospects

We tried to develop a completed account on the integration of infinite-dimensional Lie algebra to Lie 2-groups. In order to do so we dropped some topics that may be at hand, which we shortly line out in this section. Most of them deserve to be worked out seriously.

Remark V.1 (Diffeological Lie Groups). The problem that one encounters when trying to integrate central extensions of infinite-dimensional Lie algebras to Lie groups is that one has to factor out subgroups from locally convex spaces that may be indiscreet. This has to be done to ensure that the cocycle condition for a certain universal cocycle holds.

However, one may resolve this problem by enlarging the category of smooth manifolds to a category which is closed under quotients. For instance, the category of diffeological spaces (or more general smooth spaces, cf. [BH08]) has this property. From our cocycle $(F_{\omega,\beta}, \Theta_\beta)$, integrating a given Lie algebra cocycle ω, one easily constructs a cocycle $q \circ F_{\omega,\beta}$, which is in general (locally) smooth as a map between diffeological spaces. With the corresponding version Theorem II.1 for diffeological spaces one thus constructs a diffeological group $\widehat{G}_\omega$ and

$$\mathfrak{z}/\Pi_\omega \hookrightarrow \widehat{G}_\omega \twoheadrightarrow G$$

is a candidate for a central extension of diffeological groups, integrating

$$\mathfrak{z} \hookrightarrow \mathfrak{z} \oplus_\omega \mathfrak{g} \twoheadrightarrow \mathfrak{g}.$$

The crucial point here is to set up the notion of a Lie functor from diffeological spaces to vector spaces such that it takes $\mathfrak{z}/\Gamma_\omega$ to $\mathfrak{z}$, even if Γ is indiscreet (cf. [Sch08]). ∎

Remark V.2 (Differential Geometry of Generalised Extensions). One perspective to the integration procedure for central extensions of Lie algebras is to find a Lie group extension as a principal bundle with a prescribed curvature. It should be possible to develop such a point of view also for smooth generalised central extensions.

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On the level of cocycles, the passage is quite clear. For a cocycle $f : G \times G \to Z$, smooth on $U \times U$, the central extension $Z \hookrightarrow Z \times_f G \twoheadrightarrow G$ is a principal bundle, described by the transgressed[4] Čech cocycle

$$\gamma_{g,h} : gV \cap hV \to Z, \quad x \mapsto f(g, g^{-1}x) - f(h, h^{-1}x)$$

where $V \subseteq U$ is an open unit neighbourhood with $V \cdot V \subseteq U$. That $\gamma_{g,h}$ is smooth follows from

$$f(g, g^{-1}x) - f(h, h^{-1}x) = f(g^{-1}h, h^{-1}x) - f(g, g^{-1}h)$$

and from $g^{-1}h \in U$ if $gV \cap hV \neq \emptyset$. For a smooth generalised cocycle (F, Θ), the transgressed non-abelian Čech cocycle would accordingly be given by

$$\gamma_{g,h} : gV \cap hV \to Z, \quad x \mapsto F(g, g^{-1}x) - F(h, h^{-1}x) - \tau(\Theta(g, g^{-1}h, h^{-1}x))$$

and

$$\eta_{g,h,k} : gV \cap hV \cap kV \to Z,$$
$$x \mapsto -\Theta(g, g^{-1}h, h^{-1}x) - \Theta(h, h^{-1}k, k^{-1}x) + \Theta(g, g^{-1}k, k^{-1}x)$$

This yields a principal $\mathcal{Z}$-2-bundle, which should be roughly given by $\mathcal{Z} \to \mathcal{G}_{(F,\Theta)} \to \underline{G}$. The interpretation of this as a bundle with connection could be a bit more tricky, for the skeleton of this bundle should admit curvature (in fancy terms, we want the fake curvature *not* to vanish). This sits at the cutting edge development of higher bundles with connections (cf. [SW08] and references therein and [Wal08] for the case of group extensions). ∎

Remark V.3 (Non-Locally Exponential Lie Algebras). One may impose that question is whether a similar Theorem as our version for Lie's Third Theorem is also is reach for non-locally exponential Lie algebras. To our best knowledge it would be unlikely to expect a similar result in this direction, for the algebraic properties of non-locally exponential Lie algebras couple very hardly to their Lie local Lie groups (if they exist at all). For instance, Lempert proved that $\mathcal{V}(M)_{\mathbb{C}}$ is even not *integrable* for any compact manifold M (cf. [Lem97]), which relies on much more involved and less systematic arguments as the counterexample of van Est and Korthagen in [vEK64]. ∎

Remark V.4 (Higher Lie Algebras and Lie Algebroids). In a sense, we performed a similar integration procedure as in Henriques in [Hen07]. It thus seem to be promising to carry this analogy further to integrate even infinite-dimensional Lie 2-algebras or to enlarge Henriques' procedure beyond the Banach-case. In another direction, an integration procedure for infinite-dimensional Lie algebroids could also be trackable (cf. [TZ06]). ∎

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[4]this map coincides with the usual transgression map $H_{\mathrm{gp}}^n(G, Z) \to \check{H}^{n-1}(G, Z)$ if Z is discrete

Remark V.5 (Stacky Lie groups). Our definition of a Lie 2-group is some-what weaker than one would expect at first. However, is one leaves the world of manifolds and considers Lie groupoids as stacks, then we expect that our Lie 2-groups lead to stacky Lie groups in the sense of [Blo08]. It would be desirable to work out a Lie theory of stacky Lie groups in the correct categorical setup.∎

A Appendix: Differential calculus on locally convex spaces

We provide some background material on locally convex Lie groups and their Lie algebras in this appendix.

Definition A.1. Let X and Y be a locally convex spaces and $U \subseteq X$ be open. Then $f : U \to Y$ is *differentiable* or C^1 if it is continuous, for each $v \in X$ the differential quotient

$$df(x).v := \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

exists and if the map $df : U \times X \to Y$ is continuous. If $n > 1$ we inductively define f to be C^n if it is C^1 and df is C^{n-1} and to be C^∞ or *smooth* if it is C^n. We say that f is C^∞ or *smooth* if f is C^n for all $n \in \mathbb{N}_0$. We denote the corresponding spaces of maps by $C^n(U, Y)$ and $C^\infty(U, Y)$.

A (locally convex) *Lie group* is a group which is a smooth manifold modelled on a locally convex space such that the group operations are smooth. A locally convex Lie algebra is a Lie algebra, whose underlying vector space is locally convex and whose Lie bracket is continuous. ∎

Definition A.2. Let G be a locally convex Lie group. The group G is said to have an *exponential function* if for each $x \in \mathfrak{g}$ the initial value problem

$$\gamma(0) = e, \quad \gamma'(t) = T\lambda_{\gamma(t)}(e).x$$

has a solution $\gamma_x \in C^\infty(\mathbb{R}, G)$ and the function

$$\exp_G : \mathfrak{g} \to G, \quad x \mapsto \gamma_x(1)$$

is smooth. Furthermore, if there exists a zero neighbourhood $W \subseteq \mathfrak{g}$ such that $\exp_G|_W$ is a diffeomorphism onto some open unit neighbourhood of G, then G is said to be *locally exponential*. ∎

Lemma A.3. *If G and G' are locally convex Lie groups with exponential function, then for each morphism $\alpha : G \to G'$ of Lie groups and the induced morphism $d\alpha(e) : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras, the diagram*

$$
\begin{array}{ccc}
G & \xrightarrow{\ \alpha\ } & G' \\
\Big\uparrow{\scriptstyle \exp_G} & & \Big\uparrow{\scriptstyle \exp_{G'}} \\
\mathfrak{g} & \xrightarrow{\ d\alpha(e)\ } & \mathfrak{g}'
\end{array}
$$

commutes.

Remark A.4. The Fundamental Theorem of Calculus for locally convex spaces (cf. [Glö02a, Th. 1.5]) yields that a locally convex Lie group G can have at most one exponential function (cf. [Nee06, Lem. II.3.5]).

Typical examples of locally exponential Lie groups are Banach-Lie groups (by the existence of solutions of differential equations and the inverse mapping theorem, cf. [Lan99]) and groups of smooth and continuous mappings from compact manifolds into locally exponential groups ([Glö02b, Sect. 3.2], [Woc06b]). However, diffeomorphism groups of compact manifolds are never locally exponential (cf. [Nee06, Ex. II.5.13]) and direct limit Lie groups not always (cf. [Glö05, Rem. 4.7]). For a detailed treatment of locally exponential Lie groups and their structure theory we refer to [Nee06, Sect. IV]. ∎

Definition A.5. A locally convex Lie algebra $\mathfrak{g}$ is said to be locally exponential if there exists a circular convex open zero neighbourhood $U \subseteq \mathfrak{g}$ and an open subset $D \subseteq U \times U$ on which there exists a smooth map

$$m_U : D \to U, \quad (x, y) \mapsto x * y$$

such that $(D, U, m_U, 0)$ is a local Lie group and such that the following holds.

i) For $x \in U$ and $|t|, |s|, |t+s| \leq 1$, we have $(tx, sx) \in D$ with $tx * sx = (t+s)x$.

ii) The second order term in the Taylor expansion of m_U in 0 is $b(x, y) = \frac{1}{2}[x, y]$. ∎

Remark A.6. (cf. [Nee06, Ex. IV.2.4]) As above, one has that all Banach-Lie algebras are locally exponential, as well as all Lie algebras of locally exponential groups. ∎

Theorem A.7. *([Nee06, Th. IV.3.8]) Let $\mathfrak{g}$ be a locally exponential Lie algebra. Then the adjoint group $G_{\mathrm{ad}} \leq \mathrm{Aut}(\mathfrak{g})$ carries the structure of a locally exponential Lie group whose Lie algebra is $\mathfrak{g}_{\mathrm{ad}} := \mathfrak{g}/Z(\mathfrak{g})$.* ∎

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